

HASSE PRINCIPLE AND WEAK APPROXIMATION FOR MULTINORM EQUATIONS

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ABSTRACT. In this note, we are interested in local-global principles for multinorm equations of the form $\prod_{i=1}^n N_{L_i/k}(z_i) = a$ where k is a global field, L_i/k are finite separable field extensions and $a \in k^*$.

In particular, we prove a result relating weak approximation for this equation to weak approximation for some classical norm equation $N_{F/k}(w) = a$ where $F := \bigcap_{i=1}^n L_i$. It provides a proof of a "weak approximation" analogue of a recent conjecture by Pollio and Rapinchuk about multinorm principle. We also provide a counterexample to the original conjecture concerning Hasse principle.

0. INTRODUCTION

Let k be a global field, Ω be the set of places of k and $n \geq 2$. Let L_1, \dots, L_n be finite separable field extensions of k .

For any $a \in k^*$, we consider the following equation

$$\prod_{i=1}^n N_{L_i/k}(z_i) = a.$$

It defines an affine k -variety X , which is a principal homogeneous space under the k -torus T defined by the following exact sequence of k -tori

$$0 \rightarrow T \rightarrow \prod_{i=1}^n R_{L_i/k}(\mathbf{G}_m) \xrightarrow{\prod N_{L_i/k}} \mathbf{G}_m \rightarrow 0,$$

where the last map is the product of norm maps.

It is well-known that for such varieties X , the obstruction to Hasse principle is measured by the finite group $\text{III}^2(k, \widehat{T})$ and the obstruction to weak approximation by the finite group $\text{III}_{\omega}^2(k, \widehat{T}) / \text{III}^2(k, \widehat{T})$, via the Brauer-Manin obstruction (see for instance [S]), where \widehat{T} is the module of characters of T .

Recall that for any Galois module M over k , we have by definition

$$\text{III}^i(k, M) := \text{Ker} \left(H^i(k, M) \rightarrow \prod_{v \in \Omega} H^i(k_v, M) \right)$$

and

$$\text{III}_{\omega}^i(k, M) := \{ \alpha \in H^i(k, M) \text{ s.t. } \alpha_v = 0 \text{ in } H^i(k_v, M) \text{ for almost all places } v \in \Omega \}.$$

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The Hasse principle for such a variety X was studied by several authors, including Hürleimann (see [H], especially Proposition 3.3), Colliot-Thélène and Sansuc (unpublished), Platonov and Rapinchuk (see [PIR], sections 6.3 and 9.3, and in particular Proposition 6.11), Prasad and Rapinchuk (see [PrR], section 4, especially Proposition 4.2) and Pollio and Rapinchuk (see [PR], main theorem). The local-global principle for X is for instance related to some arithmetic properties of algebraic groups of type A_n (see for instance [PIR] or [PrR]).

The main result of this note (see Theorem 6) compares the defect of weak approximation for X to the defect of weak approximation for the k -variety Y defined by $N_{F/k}(w) = a$, where $F := \bigcap_{i=1}^n L_i$, under some technical assumptions. More precisely, if S denotes the norm k -torus $R_{F/k}^{(1)}(\mathbf{G}_m)$, then we prove that under some assumptions, there is a canonical isomorphism

$$\text{III}_\omega^2(k, \widehat{S}) \xrightarrow{\sim} \text{III}_\omega^2(k, \widehat{T}),$$

which essentially means that weak approximation holds on X if and only if it holds on Y . In particular, we can compute the defect of weak approximation for the multinorm equation related to (L_1, \dots, L_n) via the defect of weak approximation for the usual norm equation related to the extension F/k .

In particular, it solves an analogue for weak approximation of a conjecture by Pollio and Rapinchuk (see [PR], section 4). We also provide a counterexample for the original conjecture from [PR], which concerns the multinorm Hasse principle (see Proposition 11).

1. THE LINEARLY DISJOINT CASE

Throughout this text, we use the following classical notations : if k is field, Γ_k denotes the absolute Galois group of k . For a k -torus T , we denote by \widehat{T} the Γ_k -module of characters of T .

We start by the following case, where the Hasse principle and weak approximation hold :

Theorem 1. *Let $L_1, \dots, L_n/k$ be finite separable field extensions. Write $\{1, \dots, n\} = I \cup J$, with $I \cap J = \emptyset$ and $I, J \neq \emptyset$. Let L_I (resp. L_J) be the composite of the fields L_i , $i \in I$ (resp. $i \in J$). Define E_I (resp. E_J) to be the Galois closure of the extension L_I/k (resp. L_J/k). Define T to be the k -torus of equation $\prod_{i=1}^n N_{L_i/k}(z_i) = 1$.*

If $L_I \cap E_J = k$, then

$$\text{III}_\omega^2(k, \widehat{T}) = 0.$$

In particular, under these assumptions, for any $a \in k^$, the k -variety defined by $\prod_{i=1}^n N_{L_i/k}(z_i) = a$ satisfies the Hasse principle and weak approximation.*

Remark 2. Note that the assumption implies that $\bigcap_i L_i = k$. Note also that this result generalizes section 5 of [PR] and Corollary 2.3 of [W], by taking into account more than two field extensions. The proof is inspired by those two results.

Proof. Define M to be the k -torus $M := M_I \times M_J$ where $M_I := \prod_{i \in I} R_{L_i/k}(\mathbf{G}_m)$ (same for M_J).

Lemma 3. (i) *As a Γ_{E_J} -module (resp. as a Γ_{L_I} -module), \widehat{T} is a permutation module.*

(ii) *As a Γ_{L_I} -module, $\widehat{T} \cong \widehat{T}^{\Gamma_{E_J}} \oplus N$, where N is a permutation Γ_{L_I} -module.*

Proof. (i) We have a natural exact sequence of Γ_k -modules

$$(1) \quad 0 \rightarrow \mathbb{Z} \rightarrow \widehat{M} \rightarrow \widehat{T} \rightarrow 0.$$

As a Γ_{E_J} -module, $\widehat{M}_J \cong \mathbb{Z}^m$ is a trivial module for some integer m , therefore \widehat{T} is isomorphic to $\widehat{M}_I \oplus \mathbb{Z}^{m-1}$ as a Γ_{E_J} -module, hence it is permutation.

For any $i \in I$, we have an isomorphism of Γ_{L_i} -modules $\mathbb{Z}[L_i/k] \cong \mathbb{Z} \oplus N_i$, where N_i is a permutation module. Hence by (1), $\widehat{T} \cong \mathbb{Z}^{\#I-1} \oplus \widehat{M}$ as Γ_{L_i} -modules, where $\widehat{M} := \bigoplus_{i \in I} N_i \oplus M_J$ is a permutation Γ_{L_i} -module. Therefore \widehat{T} itself is a permutation Γ_{L_i} -module.

(ii) Consider the following commutative exact diagram of Γ_k -modules :

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{M}^{\Gamma_{E_J}} & \longrightarrow & \widehat{T}^{\Gamma_{E_J}} \longrightarrow H^1(\Gamma_{E_J}, \mathbb{Z}) = 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{T} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \widehat{M}/\widehat{M}^{\Gamma_{E_J}} & \xrightarrow{\cong} & \widehat{T}/\widehat{T}^{\Gamma_{E_J}} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}.$$

Since $L_I \cap E_J = k$, we have $\mathbb{Z}[L_i/k]^{\Gamma_{E_J}} = \mathbb{Z}[L_i/k]^{\Gamma_k} = \mathbb{Z} \cdot \varepsilon_i$ for any $i \in I$, where $\varepsilon_i := \sum_{g \in \Gamma_k/\Gamma_{L_i}} g$. We already know that $\mathbb{Z}[L_i/k] \cong \mathbb{Z} \oplus N_i$ as Γ_{L_i} -modules, therefore, since $\widehat{M}_I/\widehat{M}_I^{\Gamma_{E_J}} \cong \bigoplus_{i \in I} \mathbb{Z}[L_i/k]/\mathbb{Z} \cdot \varepsilon_i$, we deduce that the map of Γ_{L_i} -modules $\widehat{M}_I \rightarrow \widehat{M}_I/\widehat{M}_I^{\Gamma_{E_J}}$ splits, hence the map of Γ_{L_i} -modules $\widehat{M} \rightarrow \widehat{M}/\widehat{M}^{\Gamma_{E_J}} = \widehat{M}_I/\widehat{M}_I^{\Gamma_{E_J}}$ splits. Therefore the map of Γ_{L_i} -modules $\widehat{T} \rightarrow \widehat{T}/\widehat{T}^{\Gamma_{E_J}}$ also splits, which concludes the proof of the Lemma. \square

Lemma 4. *The restriction map $\rho : H^2(k, \widehat{T}) \rightarrow H^2(L_I, \widehat{T}) \oplus H^2(E_J, \widehat{T})$ is injective.*

Proof. By point (i) of Lemma 3, $H^1(E_J, \widehat{T}) = 0$. Hence the inflation-restriction exact sequence is the following

$$0 \rightarrow H^2(E_J/k, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\inf} H^2(k, \widehat{T}) \xrightarrow{\text{res}} H^2(E_J, \widehat{T}).$$

So it is enough to prove that the composite map

$$\rho' : H^2(E_J/k, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\inf} H^2(k, \widehat{T}) \xrightarrow{\text{res}'} H^2(L_I, \widehat{T})$$

is injective. But we have an isomorphism $H^2(E_J/k, \widehat{T}^{\Gamma_{E_J}}) \cong H^2(L_I \cdot E_J/L_I, \widehat{T}^{\Gamma_{E_J}})$ since we have the canonical isomorphism $\text{Gal}(E_J \cdot L_I/L_I) \xrightarrow{\cong} \text{Gal}(E_J/k)$ (because $L_I \cap E_J = k$). So we can identify the map ρ' with the following composite map

$$H^2(L_I \cdot E_J/L_I, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\inf} H^2(L_I, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\text{res}} H^2(L_I, \widehat{T}).$$

We have $H^1(L_I.E_J, \widehat{T}^{E_J}) = 0$ since \widehat{T}^{E_J} is a constant torsion-free Γ_{E_J} -module, hence the map $H^2(L_I.E_J/L_I, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\text{inf}} H^2(L_I, \widehat{T}^{\Gamma_{E_J}})$ is injective. By point (ii) of Lemma 3, we know that $\widehat{T}^{\Gamma_{E_J}}$ is a direct summand of \widehat{T} as a Γ_{L_I} -module, hence the map $H^2(L_I, \widehat{T}^{\Gamma_{E_J}}) \xrightarrow{\text{res}} H^2(L_I, \widehat{T})$ is also injective, which concludes the proof. \square

We now prove Theorem 1.

By Lemma 4, the natural restriction map

$$\text{III}_{\omega}^2(k, \widehat{T}) \rightarrow \text{III}_{\omega}^2(L_I, \widehat{T}) \oplus \text{III}_{\omega}^2(E_J, \widehat{T})$$

is injective. By point (i) of Lemma 3, we know that \widehat{T} is a permutation Γ_{L_I} -module (resp. Γ_{E_J} -module). Therefore we have $\text{III}_{\omega}^2(L_I, \widehat{T}) = \text{III}_{\omega}^2(E_J, \widehat{T}) = 0$, hence $\text{III}_{\omega}^2(k, \widehat{T}) = 0$. \square

Remark 5. • Let k be a number field and K/k be a Galois extension of group \mathbf{A}_4 (the alternating group on four elements). Let L_1 and L_2 be two different degree 4 subfields of K , hence $L_1 \cap L_2 = k$. However L_1 and L_2 are conjugate, hence

$$T \cong \text{Ker}(\mathbf{R}_{L_1/k}(\mathbf{G}_m) \times \mathbf{R}_{L_2/k}(\mathbf{G}_m) \rightarrow \mathbf{G}_m) \cong \mathbf{R}_{L_1/k}(\mathbf{G}_m) \times \mathbf{R}_{L_2/k}^{(1)}(\mathbf{G}_m).$$

We know $\text{III}_{\omega}^2(k, \widehat{\mathbf{R}_{L_1/k}^{(1)}(\mathbf{G}_m)}) = \mathbb{Z}/2\mathbb{Z}$ by Proposition 1 in [Ku] and $\text{III}_{\omega}^2(k, \widehat{\mathbf{R}_{L_2/k}(\mathbf{G}_m)}) = 0$ since it's a permutation module. Therefore $\text{III}_{\omega}^2(k, \widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $L_1 \cap L_2 = k$, hence the assumption about Galois closure is necessary for the conclusion of Theorem 1 to hold.

- Following Theorem 4.1 in [CT2] (see also [S], Remark 1.9.4), for all $a, b \in k^*$ such that $k(\sqrt{a}, \sqrt{b})/k$ is a biquadratic extension, if we define $L_1 := k(\sqrt{a})$, $L_2 := k(\sqrt{b})$ and $L_3 := k(\sqrt{ab})$, then we have $\text{III}_{\omega}^2(k, \widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $L_i \cap L_j = k$ for $1 \leq i \neq j \leq 3$. Therefore the assumption about L_i, L_j is necessary for the conclusion of Theorem 1 to hold.

2. THE GENERAL CASE

We now state the main result that deals with a more general situation when the field $\bigcap_{i=1}^n L_i$ is bigger than k . In this case, the Hasse principle or weak approximation do not hold in general, but we have the following theorem :

Theorem 6. *Let $L_1, \dots, L_n/k$ be finite separable field extensions. Define $F := \bigcap_{i=1}^n L_i$. Define T to be the k -torus of equation $\prod_{i=1}^n N_{L_i/k}(z_i) = 1$ and S to be the k -torus of equation $N_{F/k}(w) = 1$. Write $\{1, \dots, n\} = I \cup J$, with $I \cap J = \emptyset$ and $I, J \neq \emptyset$. Let F_i be a field extension of L_i such that the natural map $\text{Aut}_k(F_i) \rightarrow \text{Aut}_k(F)$ is surjective. Let F_I (resp. F_J) be the composite of the fields F_i , $i \in I$ (resp. $i \in J$). Let E_I (resp. E_J) be the Galois closure of the extension F_I/F (resp. F_J/F).*

If F/k is Galois and $F_I \cap E_J = F$, then

$$\text{III}_{\omega}^2(k, \widehat{S}) \xrightarrow{\sim} \text{III}_{\omega}^2(k, \widehat{T}).$$

This theorem implies the following corollary, which proves the "weak approximation analogue" of a conjecture by Pollio and Rapinchuk about Hasse principle for multinorm tori (see the conjecture in section 4 of [PR]) :

Corollary 7. *With the same assumptions as in the Theorem, let $a \in k^*$. Assume that the k -variety of equation $\prod_i N_{L_i/k}(z_i) = a$ has a k -point. Then weak approximation holds for the equation $\prod_i N_{L_i/k}(z_i) = a$ if it holds for the equation $N_{F/k}(w) = a$.*

We also get a particular case of their conjecture concerning the multinorm Hasse principle :

Corollary 8. *With the same assumptions as in the Theorem. Assume that the Hasse principle and weak approximation hold for equations $N_{F/k}(w) = a$ (for all $a \in k^*$). Then the Hasse principle and weak approximation holds for equations $\prod_i N_{L_i/k}(z_i) = a$ (for all $a \in k^*$).*

In particular, this corollary contains the case of two Galois extensions L_1, L_2 of k such that $L_1 \cap L_2$ is a cyclic field extension of k , case that was one motivation for the conjecture in [PR] (see the remark after the conjecture in section 4 of [PR]).

Proof. Let R' be the F -torus defined by the equation $\prod_i N_{L_i/F}(z_i) = 1$ and $R := R_{F/k}(R')$.

We have an exact sequence of k -tori :

$$0 \rightarrow R \rightarrow T \rightarrow S \rightarrow 0,$$

where the morphism $T \rightarrow S$ is given by $w = \prod_i N_{L_i/F}(z_i)$.

The dual exact sequence of Galois modules

$$0 \rightarrow \widehat{S} \rightarrow \widehat{T} \rightarrow \widehat{R} \rightarrow 0$$

induces a long exact sequence

$$H^1(k, \widehat{R}) \rightarrow H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T}) \rightarrow H^2(k, \widehat{R}).$$

We know that $H^i(k, \widehat{R}) = H^i(F, \widehat{R}')$.

We first prove that $\text{III}_\omega^2(k, \widehat{R}) = 0$. This is a consequence of the canonical isomorphism $\text{III}_\omega^2(k, \widehat{R}) \cong \text{III}_\omega^2(F, \widehat{R}')$, and of the fact that $\text{III}_\omega^2(F, \widehat{R}') = 0$ (see Theorem 1; see also [PR], section 5 or [W], corollary 2.3 in some particular cases). Therefore $\text{III}_\omega^2(k, \widehat{T})$ is contained in the image of the map $H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T})$.

Let us prove now that the map $H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T})$ is injective. We have

$$H^1(k, \widehat{R}) = H^1(F, \widehat{R}') = \text{Ker}(H^2(F, \mathbb{Z}) \rightarrow \bigoplus_i H^2(L_i, \mathbb{Z})).$$

Since $F = \bigcap_i L_i$, we know that the map $H^2(F, \mathbb{Z}) \rightarrow \bigoplus_i H^2(L_i, \mathbb{Z})$ is injective, therefore $H^1(k, \widehat{R}) = 0$, hence the map $H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T})$ is injective.

Now we start to prove that the kernel of the map $\psi : H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T}) / \text{III}_\omega^2(k, \widehat{T})$ is equal to $\text{III}_\omega^2(k, \widehat{S})$. This kernel of ψ clearly contains $\text{III}_\omega^2(k, \widehat{S})$. Let us prove the converse inclusion.

First we show that $\text{III}_\omega^2(F, \widehat{T}_F) = 0$.

Since F/k is Galois, $L_i \otimes_k F \cong \prod_{\sigma \in \text{Gal}(F/k)} L_i^\sigma \cong \prod_{\sigma \in \text{Gal}(F/k)} \tilde{\sigma}(L_i)$ as F -algebras, where L_i^σ denotes the field L_i endowed with the F -algebra structure given by the morphism $F \xrightarrow{\sigma} F \rightarrow L_i$, and $\tilde{\sigma} \in \text{Aut}_k(F_i)$ is a (chosen) lift of σ since the natural map $\text{Aut}_k(F_i) \rightarrow \text{Gal}(F/k)$ is surjective.

Let \tilde{L}_I (resp. \tilde{L}_J) be the composite of the fields $\tilde{\sigma}(L_i)$ where $\sigma \in \text{Gal}(F/k)$ and $i \in I$ (resp. $i \in J$). Since $\tilde{\sigma}(L_i) \subset F_i$, it implies $\tilde{L}_I \subset F_I$ and $\tilde{L}_J \subset F_J$. Since $F_I \cap F_J = F$, it implies that $\tilde{L}_I \cap \tilde{L}_J = F$, hence $\text{III}_\omega^2(F, \widehat{T}_F) = 0$ by Theorem 1.

Consider the following commutative diagram

$$\begin{array}{ccc} H^2(k, \widehat{S}) & \xrightarrow{\psi} & H^2(k, \widehat{T}) / \text{III}_\omega^2(k, \widehat{T}) \\ \downarrow & & \downarrow \\ H^2(F, \widehat{S}_F) & \xrightarrow{\psi_F} & H^2(F, \widehat{T}_F) / \text{III}_\omega^2(F, \widehat{T}_F) = H^2(F, \widehat{T}_F). \end{array}$$

In the following, we will show $H^1(F, \widehat{R}) = 0$, hence the map ψ_F in the diagram is injective. Note that by definition R_F is identified with the kernel of the product of norm maps

$$\prod_{i=1}^n R_{L_i \otimes_k F / F} \mathbb{G}_m \rightarrow R_{F \otimes_k F / F} \mathbb{G}_m,$$

hence we get an isomorphism

$$H^1(F, \widehat{R}) = \bigoplus_{\sigma \in \text{Gal}(F/k)} \text{Ker} \left(H^2(F, \mathbb{Z}) \rightarrow \bigoplus_i H^2(\tilde{\sigma}(L_i), \mathbb{Z}) \right).$$

Since $F_I \cap E_J = F$, it implies that $F = \bigcap_i \tilde{\sigma}(L_i)$ for all $\sigma \in \text{Gal}(F/k)$. Therefore the map $H^2(F, \mathbb{Z}) \rightarrow \bigoplus_i H^2(\tilde{\sigma}(L_i), \mathbb{Z})$ is injective for all $\sigma \in \text{Gal}(F/k)$, hence $H^1(F, \widehat{R}) = 0$.

Let $\alpha \in \text{Ker}(\psi)$. Then $\alpha_F = 0$ in $H^2(F, \widehat{S}_F)$. Hence $\alpha \in H^2(F/k, \widehat{S})$. Considering the long exact sequence associated to the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[F/k] \rightarrow \widehat{S} \rightarrow 0$$

we get an isomorphism $H^2(F/k, \widehat{S}) \xrightarrow{\sim} H^3(F/k, \mathbb{Z})$. For any $g \in \text{Gal}(F/k)$, we have $H^3(\langle g \rangle, \mathbb{Z}) = 0$, hence we get that $H^2(F/k, \widehat{S}) \subset \text{III}_\omega^2(k, \widehat{S})$ (in fact, this is an equality). Hence $\alpha \in \text{III}_\omega^2(k, \widehat{S})$, i.e. $\text{Ker}(\psi) \subset \text{III}_\omega^2(k, \widehat{S})$.

It concludes the proof : the map $H^2(k, \widehat{S}) \rightarrow H^2(k, \widehat{T})$ induces an isomorphism

$$\text{III}_\omega^2(k, \widehat{S}) \cong \text{III}_\omega^2(k, \widehat{T}).$$

□

Example 9. (i) The assumptions of Theorem 6 hold if $n = 2$ and $L_1, L_2/k$ are Galois and $L_1 \cap L_2 = F$.

(ii) They also hold if $n \geq 2$, $L_1, \dots, L_n/k$ are Galois and $(L_1 \dots L_r) \cap (L_{r+1} \dots L_n) = F$ (for some $1 \leq r < n$).

(iii) Let L_I (resp. L_J) be the composite of the fields L_i , $i \in I$ (resp. $i \in J$). Let \tilde{E}_I (resp. \tilde{E}_J) be the Galois closure of the extension L_I/k (resp. L_J/k). The assumptions of Theorem 6 also hold if $\tilde{E}_I \cap \tilde{E}_J = F$.

(iv) Let $k = \mathbb{Q}$ and $L_1 := \mathbb{Q}(\sqrt{2}, \sqrt[4]{3})$ and $L_2 := \mathbb{Q}(\sqrt[4]{2})$. Then $F = L_1 \cap L_2 = \mathbb{Q}(\sqrt{2})$. We can choose $F_1 = L_1$ and $F_2 = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$. Since $E_2 = F_2$, it implies $F_1 \cap E_2 = F$. Therefore we get $\text{III}_\omega^2(\widehat{T}) \cong \text{III}_\omega^2(k, \widehat{S}) = 0$ (since F/\mathbb{Q} is cyclic) while $\tilde{E}_1 \cap \tilde{E}_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}) \neq F$ since $\tilde{E}_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt[4]{3})$ and $\tilde{E}_1 = E_2$. Therefore Theorem 6 is more general than point (iii).

The following example shows that even in the case of two field extensions, some assumptions about Galois closures have to be made for Theorem 6 to hold:

Example 10. Let k be a number field, $a \in k$ and $b, d \in k^*$. Let $L_1 = k(\sqrt{a - b\sqrt{d}})$ be a field extension of k of degree 4. Suppose $m := a^2 - b^2d$ is not a square in $k(\sqrt{d})$ (eg. $k = \mathbb{Q}$ and $(a, b, d, m) = (1, 1, 2, -1)$), hence L_1/k is non-Galois. Let $L_2 = k(\sqrt{d}, \sqrt{m})$. Then $F = L_1 \cap L_2 = k(\sqrt{d})$ and

$$\text{III}_\omega^2(k, \widehat{T}) = \mathbb{Z}/2\mathbb{Z} \text{ while } \text{III}_\omega^2(k, \widehat{S}) = 0.$$

Proof. Note that $F_1 \supset L_1(\sqrt{m})$, hence $F_1 \cap F_2 \supset F_1 \cap L_2 = L_2 \neq F$, hence the assumptions of Theorem 6 don't satisfy.

Let $G := \text{Gal}(L/k) \cong \mathbf{D}_4$ where $L := L_1.L_2$ is the Galois closure of L_1/k , let \widehat{M} be the permutation G -module $\mathbb{Z}[L_2/k]$ and $\widehat{N} := \mathbb{Z}[L_1/k]/\mathbb{Z}$. Then we have an exact sequence of G -modules

$$0 \rightarrow \widehat{M} \rightarrow \widehat{T} \rightarrow \widehat{N} \rightarrow 0,$$

hence an exact sequence of cohomology groups :

$$0 = H^1(G, \widehat{M}) \rightarrow H^1(G, \widehat{T}) \xrightarrow{\rho} H^1(G, \widehat{N}) \rightarrow H^2(G, \widehat{M}) \rightarrow H^2(G, \widehat{T}) \rightarrow H^2(G, \widehat{N}).$$

But we have isomorphisms of finite groups

$$H^1(G, \widehat{T}) \cong \text{Ker}(H^1(L/k, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(L/L_1, \mathbb{Q}/\mathbb{Z}) \oplus H^1(L/L_2, \mathbb{Q}/\mathbb{Z})) = H^1(F/k, \mathbb{Q}/\mathbb{Z})$$

and

$$H^1(G, \widehat{N}) \cong \text{Ker}(H^1(L/k, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(L/L_1, \mathbb{Q}/\mathbb{Z})) = H^1(F/k, \mathbb{Q}/\mathbb{Z}),$$

therefore ρ is an isomorphism. Hence we get a commutative diagram with exact rows

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^2(G, \widehat{M}) & \longrightarrow & H^2(G, \widehat{T}) & \longrightarrow & H^2(G, \widehat{N}) \\ & & \downarrow \phi = (\phi_g) & & \downarrow & & \downarrow \\ & & \prod_{g \in G} H^2(\langle g \rangle, \widehat{M}) & \xrightarrow{\psi = (\psi_g)} & \prod_{g \in G} H^2(\langle g \rangle, \widehat{T}) & \longrightarrow & \prod_{g \in G} H^2(\langle g \rangle, \widehat{N}). \end{array}$$

Since $G \cong \mathbf{D}_4$ and L is the Galois closure of L_1/k , Proposition 1 in [Ku] implies that $\text{III}_\omega^2(k, \widehat{N}) = 0$. So we deduce from an easy diagram chase in (2) that

$$\text{Ker}(\psi \circ \phi) \xrightarrow{\sim} \text{III}_\omega^2(k, \widehat{T}).$$

We now prove that $\psi \circ \phi = 0$. For any $g \in G$, let L^g denote the subfield of L fixed by g . We consider the two following cases :

- (i) if $g \notin \text{Gal}(L/F')$, where $F' := k(\sqrt{m})$. Then g has order 2 (since $G \cong \mathbf{D}_4$), and $L^g \neq L_2$. Then there exists a quadratic extension K'/k contained in L_2 such that $L^g.K' = L$. Therefore $L^g \otimes_k L_2 = L \otimes_{K'} L_2 = L \oplus L$. Therefore

$$H^2(\langle g \rangle, \widehat{M}) = H^2(L/L^g, \mathbb{Z}[L_2/k]) = H^2(L/L^g, \mathbb{Z}^2[L/L^g]) = 0,$$

hence $\phi_g = 0$ and in particular $\psi_g \circ \phi_g = 0$.

- (ii) if $g \in \text{Gal}(L/F') \cong \mathbb{Z}/4\mathbb{Z}$, then one checks easily that it is enough to consider the case $\langle g \rangle = \text{Gal}(L/F')$. We now assume that g has order 4. Let $\widehat{M}' := \mathbb{Z}[L_2/k]/\mathbb{Z}$ and $\widehat{N}' := \mathbb{Z}[L_1/k]$. We have a natural exact sequence :

$$0 \rightarrow \widehat{N}' \rightarrow \widehat{T} \rightarrow \widehat{M}' \rightarrow 0,$$

hence an exact sequence

$$H^2(L/F', \widehat{N}') \rightarrow H^2(L/F', \widehat{T}) \rightarrow H^2(L/F', \widehat{M}').$$

But we have $H^2(L/F', \widehat{N}') = H^2(L/F', \mathbb{Z}[L_1/k]) = 0$ since $F' \cdot L_1 = L$. Hence $H^2(L/F', \widehat{T}) \rightarrow H^2(L/F', \widehat{M}')$ is injective. Therefore it is enough to prove that the composite map

$$H^2(G, \mathbb{Z}[L_2/k]) \xrightarrow{\phi_g} H^2(L/F', \mathbb{Z}[L_2/k]) \xrightarrow{\psi'_g} H^2(L/F', \widehat{M}')$$

is the zero map. But ψ'_g factors through the cokernel of the natural map $i_g : H^2(L/F', \mathbb{Z}) \rightarrow H^2(L/F', \mathbb{Z}[L_2/k])$, hence we only need to prove that the composite map

$$H^2(G, \mathbb{Z}[L_2/k]) \xrightarrow{\phi_g} H^2(L/F', \mathbb{Z}[L_2/k]) \xrightarrow{\psi''_g} \text{Coker}(i_g)$$

is zero.

The image of ϕ_g in $H^2(L/F', \mathbb{Z}[L_2/k])$ is $\text{Gal}(F'/k)$ -invariant, hence we only need to show that ψ''_g restricted to $H^2(L/F', \mathbb{Z}[L_2/k])^{\text{Gal}(F'/k)}$ is zero. Since $\mathbb{Z}[L_2/k] \cong \mathbb{Z}[F/k] \otimes \mathbb{Z}[F'/k]$ canonically as G -modules, it implies that

$$H^2(L/F', \mathbb{Z}[L_2/k]) \cong H^2(L/F', \mathbb{Z}[F/k]) \otimes \mathbb{Z}[F'/k]$$

as $\text{Gal}(F'/k)$ -modules. Let σ is the unique nontrivial element of $\text{Gal}(F'/k)$, then σ induces an isomorphism $H^2(L/F', \mathbb{Z}[F/k]) \rightarrow H^2(L/F', \mathbb{Z}[F/k]), \chi \mapsto \chi^\sigma$. Let

$$(\chi_1, \chi_2) \in \prod_{\text{Gal}(F'/k)} H^2(L/F', \mathbb{Z}[F/k]) \cong H^2(L/F', \mathbb{Z}[F/k]) \otimes \mathbb{Z}[F'/k].$$

Then $\sigma(\chi_1, \chi_2) = (\chi_2^\sigma, \chi_1^\sigma)$ by the definition of the action of $\text{Gal}(F'/k)$ on $H^2(L/F', \mathbb{Z}[L_2/k])$. Since $H^2(L/F', \mathbb{Z}[F/k]) \cong H^2(L/L_2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, we have $\chi_i^\sigma = \chi_i$ for $i = 1, 2$. Therefore

$$H^2(L/F', \mathbb{Z}[L_2/k])^{\text{Gal}(F'/k)} = \{(\chi, \chi) \mid \chi \in H^2(L/F', \mathbb{Z}[F/k])\},$$

hence it is just the image of i_g , so its image by ψ''_g is zero.

Then we deduce that $\psi''_g \circ \phi_g = 0$, hence $\psi_g \circ \phi_g = 0$.

So we proved that $\psi \circ \phi = 0$, hence we get

$$\text{III}_\omega^2(k, \widehat{T}) = H^2(G, \widehat{M}) \cong H^2(L/L_2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

which concludes the proof. \square

3. A COUNTEREXAMPLE TO THE ORIGINAL CONJECTURE

We now construct a counterexample to the original conjecture of Pollio and Rapinchuk, relative to the multinorm Hasse principle :

Proposition 11. *Let $k = \mathbb{Q}$. Let $q = 2$ or $q \equiv 5 \pmod{8}$ be a prime. Suppose m is an integer and*

$$m \equiv \begin{cases} \pm 1 \pmod{8} \text{ or } \pm 2 \pmod{16}, & \text{if } q = 2, \\ \pm 1, \pm 5 \pmod{8}, & \text{if } q \equiv 5 \pmod{8}, \end{cases}$$

and none of $\pm m, \pm mq$ is a square in \mathbb{Q}^ (eg. $(q, m) = (2, 7)$ or $(5, 17)$). Let $L_1 = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{q})$ and $L_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{m\sqrt{q}})$. Then $F = L_1 \cap L_2 = \mathbb{Q}(\sqrt{-1}, \sqrt{q})$, $\text{III}^2(\mathbb{Q}, \widehat{T}) = \mathbb{Z}/2\mathbb{Z}$ while $\text{III}^2(\mathbb{Q}, \widehat{S}) = 0$.*

In particular, any subextension of F/\mathbb{Q} satisfies the Hasse norm principle, but $(L_1, L_2; \mathbb{Q})$ does not satisfy the multinorm principle, i.e. there exists $a \in \mathbb{Q}^$*

such that the equation $N_{L_1/\mathbb{Q}}(z_1) \cdot N_{L_2/\mathbb{Q}}(z_2) = a$ violates Hasse principle, while the equation $N_{F/\mathbb{Q}}(w) = a$ has a rational solution.

Remark 12. Geometrically, the situation is the following : the equation $N_{L_1/k}(z_1) \cdot N_{L_2/k}(z_2) = a$ defines a k -torsor Y under T . One can take the quotient of Y by the k -subtorus R (see proof of Theorem 6) : the k -variety $X := Y/R$ is a k -torsor under S , and its equation is $N_{F/k}(w) = a$. The quotient map $\pi : Y \rightarrow X$ is a torsor under R , and we know that k -torsors under R satisfy Hasse principle. In general, for a fibration $\pi : Y \rightarrow X$, one cannot deduce Hasse principle on Y from Hasse principle of the basis X and Hasse principle on rational fibers of π . But by classical fibration methods, one can sometimes prove that Hasse principle holds on Y assuming that X satisfies the Hasse principle and weak approximation (see for instance [CT1], section 3). And indeed, in the multinorm situation Proposition 11 shows that the fibration method and Hasse principle on Y fail due to the failure of weak approximation on X , while Theorem 6 implies that under the stronger assumption that X do satisfy the Hasse principle and weak approximation, the variety Y do satisfy Hasse principle.

Proof. Both L_1 and L_2 are Galois over \mathbb{Q} . Since $\pm m, \pm mq$ are not squares in \mathbb{Q}^* , then $L_1 \neq L_2$.

First, Sansuc proved that $\text{III}_\omega^2(\mathbb{Q}, \widehat{S}) = \mathbb{Z}/2\mathbb{Z}$ (see [S], (2.16)). Hence by Theorem 6, we get $\text{III}_\omega^2(\mathbb{Q}, \widehat{T}) = \mathbb{Z}/2\mathbb{Z}$. Similarly, we know that in this case $\text{III}^2(\mathbb{Q}, \widehat{S}) = 0$ since $\text{Gal}(F/\mathbb{Q}) = \text{Gal}(F_2/\mathbb{Q}_2)$, where $F_2 = F \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is a field.

By theorem 6, we have the following commutative diagram

$$\begin{array}{ccc} H^2(F/\mathbb{Q}, \widehat{S}) = \text{III}_\omega^2(\mathbb{Q}, \widehat{S}) & \xrightarrow{\cong} & \text{III}_\omega^2(\mathbb{Q}, \widehat{T}) \\ \downarrow f & & \downarrow \\ \prod_p H^2(F_p/\mathbb{Q}_p, \widehat{S}) & \xrightarrow{g=(g_p)} & \prod_p H^2(\mathbb{Q}_p, \widehat{T}), \end{array}$$

where $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. This implies that $\text{III}^2(k, \widehat{T}) \cong \text{Ker}(g \circ f)$.

If $p \neq 2$, the F_p is a product of cyclic field extensions of \mathbb{Q}_p , therefore $H^3(F_p/\mathbb{Q}_p, \mathbb{Z}) = 0$. And we know that $H^2(F_p/\mathbb{Q}_p, \widehat{S}) \cong H^3(F_p/\mathbb{Q}_p, \mathbb{Z})$, therefore for all odd p , $H^2(F_p/\mathbb{Q}_p, \widehat{S}) = 0$.

Hence $\text{III}^2(k, \widehat{T}) \cong \text{Ker}(g_2 \circ f)$. Therefore, we only need to prove that $g_2 = 0$.

Since m is a square in $\mathbb{Q}_2(\sqrt{-1}, \sqrt{q})$, then $L_1 \otimes_{\mathbb{Q}} \mathbb{Q}_2 = L_2 \otimes_{\mathbb{Q}} \mathbb{Q}_2$. Denote this degree 8 field extension of \mathbb{Q}_2 by L_v . Note that L_v/\mathbb{Q}_2 is a degree 8 Galois extension, with Galois group isomorphic to the dihedral group \mathbf{D}_4 .

We deduce that $T_2 := T \times_{\mathbb{Q}} \mathbb{Q}_2$ is isomorphic as a \mathbb{Q}_2 -torus to the torus defined by the equation $N_{L_v/\mathbb{Q}_2}(w_1) \cdot N_{L_v/\mathbb{Q}_2}(w_2) = 1$, so $T_2 \cong \mathbf{R}_{L_v/\mathbb{Q}_2}(\mathbf{G}_m) \times \mathbf{R}_{L_v/\mathbb{Q}_2}^{(1)}(\mathbf{G}_m)$. Let $T' := \mathbf{R}_{L_v/\mathbb{Q}_2}^{(1)}(\mathbf{G}_m)$. Then the natural map $T_2 \rightarrow S_2$ factors through T' , hence we only need to prove that the map $h : H^2(F_2/\mathbb{Q}_2, \widehat{S}_2) \rightarrow H^2(L_v/\mathbb{Q}_2, \widehat{T}')$ is zero.

Define $G := \text{Gal}(L_v/\mathbb{Q}_2) \cong \mathbf{D}_4$, $H := \text{Gal}(L_v/F_2) \cong \mathbb{Z}/2\mathbb{Z}$, so that $G/H \cong$

$\text{Gal}(F_2/\mathbb{Q}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have an commutative exact diagram of G -modules :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}[G/H] & \longrightarrow & \widehat{S}_2 \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \widehat{T}' \longrightarrow 0 \end{array}$$

that induces the following commutative diagram

$$\begin{array}{ccc} H^2(G/H, \widehat{S}_2) & \xrightarrow{\cong} & H^3(G/H, \mathbb{Z}) \\ \downarrow h & & \downarrow \text{inf} \\ H^2(G, \widehat{T}') & \xrightarrow{\cong} & H^3(G, \mathbb{Z}). \end{array}$$

Therefore we only need to prove that the inflation map $H^3(G/H, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$, i.e. the inflation map $H^2(G/H, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$, is zero. Consider the restriction-inflation exact sequence

$$(3) \quad H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{Res}} H^1(H, \mathbb{Q}/\mathbb{Z})^{G/H} \xrightarrow{\delta} H^2(G/H, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{inf}} H^2(G, \mathbb{Q}/\mathbb{Z}).$$

Since H is exactly the derived subgroup of $G \cong \mathbf{D}_4$, the restriction map $H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{Res}} H^1(H, \mathbb{Q}/\mathbb{Z})^{G/H}$ is the zero map. So the map δ is injective. Moreover, $H \cong \mathbb{Z}/2\mathbb{Z}$ and H is central in G , therefore $H^1(H, \mathbb{Q}/\mathbb{Z})^{G/H} \cong \mathbb{Z}/2\mathbb{Z}$. A classical computation of Schur (see for instance [Ka], corollary 2.2.12) implies that

$$H^2(G/H, \mathbb{Q}/\mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

hence the map δ is surjective.

Eventually, the exact sequence (3) implies that the map inf is zero, which concludes the proof. \square

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REFERENCES

- [CT1] J.-L. Colliot-Thélène, *L'arithmétique des variétés rationnelles*, Ann. Fac. Sci. Toulouse Math. (6) **3** (1992), 295–336.
- [CT2] J.-L. Colliot-Thélène, *Groupe de Brauer non ramifié d'espaces homogènes de tores*, preprint (2012).
- [H] W. Hürlimann, *On algebraic tori of norm type*, Comment. Math. Helv., **59** (1984) 539–549.
- [Ka] G. Karpilovsky, *The Schur Multiplier*, London Mathematical Society Monographs 2. New-York: Oxford University Press (1987).
- [Ku] B. È. Kunyavskii. Arithmetic properties of three-dimensional algebraic tori. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 116:102–107, 163, 1982. Integral lattices and finite linear groups.
- [PIR] V. Platonov and A.S. Rapinchuk, *Algebraic groups and number theory*, **139**, Academic Press Inc. (1994).
- [PR] T. Pollino and A.S. Rapinchuk, *The multinorm principle for linearly disjoint Galois extensions*, Journal of Number Theory **133** (2013), 802–821.

- [PrR] G. Prasad and A.S. Rapinchuk, *Local-global principles for embedding of fields with involution into simple algebras with involution*, Comment. Math. Helv., **85** (2010) 583–645.
- [S] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. **327** (1981), 12–80.
- [W] D. Wei, *On the equation $P(t) = N_{K/k}(\Xi)$* , preprint (2012).

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